Optimal designs for the Kalman filter applied to electrical power distribution grids

Christine Müller,

Department of Statistics (TU Dortmund)

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State space model

$(V_t)_{t\in[0,T]}$ state process voltage values at N nodes,
$(Z_t)_{t\in[0,T]}$ observed process of power values at N nodes.
General model:

$$\begin{aligned} V_t = (e_1^t, f_1^t, \ldots, e_N^t, f_N^t)^\top &= \Theta_v V_{t-1} + \Theta_x x_t + E_t^v, \;\; E_t^v \sim \mathcal{N}(0_{2N}, \Sigma^x) \\ Z_t &= h(V_t) + E_t^z, \;\; E_t^z \sim \mathcal{N}(0_{2N}, \Sigma^z) \end{aligned}$$

where $\Theta_v \in \mathbb{R}^{2N\times 2N}$, $\Theta_x \in \mathbb{R}^{2N\times D}$ effect of D explanatory variables,

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where $\Theta_v \in \mathbb{R}^{2N\times 2N}$, $\Theta_x \in \mathbb{R}^{2N\times D}$ effect of D explanatory variables,

$$h(V_t) = (Re(\underline{S_1^t}), Im(\underline{S_1^t}), \dots, Re(\underline{S_N^t}), Im(\underline{S_N^t}))^\top \in \mathbb{R}^{2N}$$

with

$$\underline{S_n^t} = P_n^t + jQ_n^t = (e_n^t + j\cdot f_n^t)\cdot \sum_{k=1}^{N+1} \underline{a_{nk}}^\star (e_k^t + j\cdot f_k^t)^\star$$

$\underline{a_{nk}}$ (n,k)'th component of known admittance matrix A,

j imaginary unit, $\underline{z}^\star$ complex conjugate of $z \in \mathbb{C}$, $e_{N+1}^t = 1$, $f_{N+1}^t = 0$.

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Design problem:

$$
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\end{aligned}
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$\Sigma^z = \text{diag}(\sigma_1^2, \sigma_1^2, \ldots, \sigma_N^2, \sigma_N^2) = \Sigma(d) = \text{diag}(\frac{1}{\delta_1}, \frac{1}{\delta_1}, \ldots, \frac{1}{\delta_N}, \frac{1}{\delta_N})$
with $d = (\delta_1, \ldots, \delta_N)$,
$\delta_n = \frac{1}{\sigma_n^2}$ precision of the measurements at node n.

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Optimal design:

$$d_c^* \in \arg\min\{C(d);\ d \in \Delta_c\},$$

where $C(d)$ is design criterion from Kalman filter depending on $\Sigma(d)$.

$$V_t = \Theta_v V_{t-1} + \Theta_x x_t + E_t^v, \quad E_t^v \sim \mathcal{N}(0_{2N}, \Sigma^v),$$
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a) **Prediction of** $V_t | z_{1:t-1} \sim \mathcal{N}(a_t, R_t)$

$$a_t = \Theta_v c_{t-1} + \Theta_x x_t, \qquad R_t = \Theta_v C_{t-1} \Theta_v^\top + \Sigma^v =: C_{1,t}(d).$$

b) **Prediction of** $Z_t | z_{1:t-1} \sim \mathcal{N}(b_t, S_t)$

$$b_t \approx h(a_t), \qquad S_t \approx h'(a_t) R_t \, h'(a_t)^\top + \Sigma(d) =: C_{2,t}(d).$$

c) **Update/Estimation of** $V_t | z_{1:t} \sim \mathcal{N}(c_t, C_t)$

$$\begin{aligned} c_t &= a_t + R_t \, h'(a_t)^\top S_t^{-1} (z_t - h(a_t)), \\ C_t &= R_t - R_t \, h'(a_t)^\top S_t^{-1} h'(a_t) R_t =: C_{3,t}(d). \end{aligned}$$

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Kalman filter

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$$\Longrightarrow C_{0,t}(d) := h'(a_t)^{-1}\Sigma(d) \, h'(a_t)^{-\top}.$$

Design criteria for Kalman filter

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\begin{aligned}
C_{0,t}(d) &:= h'(a_t)^{-1}\Sigma(d)\, h'(a_t)^{-\top}, \\
C_{1,t}(d) &= R_t = R_t(d) = \Theta_v C_{3,t-1}(d)\Theta_v^\top + \Sigma^v \quad \text{(Cov for voltage prediction)}, \\
C_{2,t}(d) &= h'(a_t)[R_t(d) + C_{0,t}(d)]\ h'(a_t)^\top \quad \text{(Cov for power prediction)}, \\
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Theorem 1

$C_{0,t}$ minimized $\forall t \Longrightarrow C_{1,t}$, $C_{2,t}$, $C_{3,t}$ minimized $\forall t$ (in Löwner ordering).

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$$d_1 := \left(\tfrac{c}{N}, \ldots, \tfrac{c}{N}\right) \in \arg\min\{\det(C_{0,t}(d);\; d \in \Delta_c\}.$$

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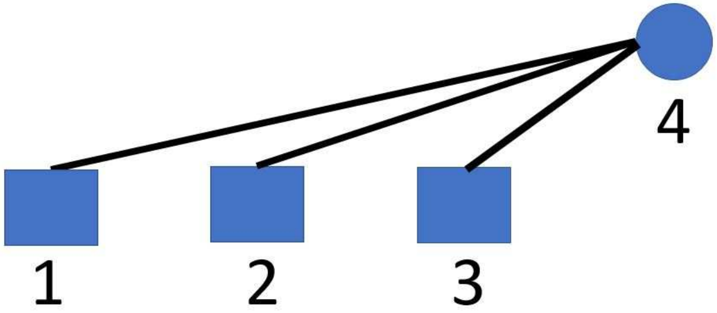
What about

tr$(C_{0,t}(d)$, tr$(C_{1,t}(d)$, tr$(C_{2,t}(d)$, tr$(C_{3,t}(d)$,

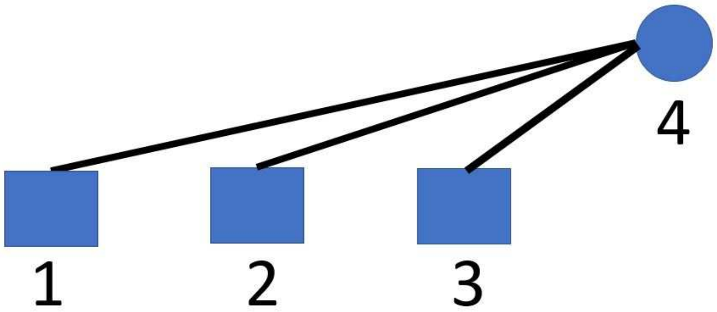
$\det(C_{1,t}(d)$, $\det(C_{2,t}(d)$, $\det(C_{3,t}(d)$?

Star network



$$\color{red}{C_{0,t}(d)} \quad := \quad h'(a_t)^{-1}\color{red}{\Sigma(d)}\, h'(a_t)^{-\top},$$
$$\color{red}{C_0(d)} := h'(v)^{-1}\color{red}{\Sigma(d)}\, h'(v)^{-\top}.$$

Star network



$$\color{red}{C_{0,t}(d)} := h'(a_t)^{-1}\color{red}{\Sigma(d)}\color{black}\, h'(a_t)^{-\top},$$
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$$h'(v) \quad = \quad h'(e_1, f_1, \ldots, e_N, f_N) = \frac{\partial h}{\partial v} = \begin{pmatrix} A_{11} & 0_{2\times 2} & \ldots & 0_{2\times 2} \\ 0_{2\times 2} & A_{22} & & 0_{2\times 2} \\ \vdots & & \ddots & \\ 0_{2\times 2} & 0_{2\times 2} & & A_{II} \end{pmatrix}.$$

$$A_{nn} \quad = \quad \begin{pmatrix} g_{nn}\,(2\,e_n - 1) & 2\,f_n\,g_{nn} - b_{nn} \\ -b_{nn}\,(2\,e_n - 1) & -2\,f_n\,b_{nn} - g_{nn} \end{pmatrix}, \quad \underline{a_{nn}} = g_{nn} + j\,b_{nn}.$$

$$\Rightarrow \qquad \mathrm{tr}(A_{nn}^{-1}A_{nn}^{-\top}) = \frac{w_n^2}{|\underline{a_{nn}}|^2} \;\text{ with }\; w_n^2 := \frac{(2e_n - 1)^2 + 4f_n^2 + 1}{(2e_n - 1)^2}.$$

A-optimal designs for star network

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$C_0(d) := h'(v)^{-1}\Sigma(d)\, h'(v)^{-\top}$, $w_n^2 := \frac{(2e_n-1)^2+4f_n^2+1}{(2e_n-1)^2}$, $n = 1, \ldots, N$.

Theorem 3. For the fixed reference state $v = (e_1, f_1, \ldots, e_N, f_N)^\top$, we get

$d_0 := d^*_{\mathrm{tr}(C_0)} = (\delta_1^*, \ldots, \delta_N^*) \in \arg\min\{\mathrm{tr}(C_0(d);\ d \in \Delta_c\}$

with $\frac{1}{\delta_n^*} = \sigma_n^{2*} = \frac{\underline{|a_{nn}|}}{w_n \cdot c} \sum_{k=1}^{N} \frac{w_k}{\underline{|a_{kk}|}}$.

A-optimal designs for star network

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E.g. $w_n = 2$ if $e_n = 1$, $f_n = 0$, for $n = 1, \ldots, N$.

A-optimal designs for star network

$$C_0(d) := h'(a_t^*)^{-1}\Sigma(d)\, h'(a_t^*)^{-\top},$$
$$C_{3,t}(d) = R_t(d) - R_t(d)[R_t(d) + C_{0,t}(d)]^{-1}\ R_t(d)$$

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Theorem 4. Assume $e_n^* = 1$, $f_n^* = 0$,

$R_t(d) = R = \text{Diag}(\lambda_1 I\!I_{2\times 2}, \dots, \lambda_N I\!I_{2\times 2})$, so that

$\overline{C}_3(d) := R - R[R + C_0(d)]^{-1}\ R$

Let $c \geq \left(\max_{k=1,\dots,N} \frac{1}{\lambda_k\,|\underline{a_{kk}}|}\right) \sum_{k=1}^N \frac{1}{|\underline{a_{kk}}|} - \sum_{k=1}^N \frac{1}{\lambda_k\,|\underline{a_{kk}}|^2}$.

a)

$$d_* = (\delta_1^*, \dots, \delta_N^*) \in \arg\min\{\text{tr}(\overline{C}_3(d));\ d \in \Delta_c\}$$

$$\Longleftrightarrow$$

$$\delta_n^* = \delta_n^{*c} := \frac{c + \sum_{k=1}^N \frac{1}{\lambda_k\,|\underline{a_{kk}}|^2}}{\sum_{k=1}^N \frac{1}{|\underline{a_{kk}}|}} \frac{1}{|\underline{a_{nn}}|} - \frac{1}{\lambda_n\,|\underline{a_{nn}}|^2}.$$

b) $\delta_n^{*c} = 0 \Longleftrightarrow c = \frac{1}{\lambda_n\,|\underline{a_{nn}}|} \sum_{k=1}^N \frac{1}{|\underline{a_{kk}}|} - \sum_{k=1}^N \frac{1}{\lambda_k\,|\underline{a_{kk}}|^2}$.

c) $\lim_{c\to\infty} \frac{\delta_k^{*c}}{\delta_n^{*c}} = \frac{|\underline{a_{nn}}|}{|\underline{a_{kk}}|}$ for all $n, k = 1, \dots, N$.

D-optimal designs for star network

$C_0(d) := h'(a_t^*)^{-1}\Sigma(d)\, h'(a_t^*)^{-\top}$,

$C_{3,t}(d) = R_t(d) - R_t(d)[R_t(d) + C_{0,t}(d)]^{-1}\ R_t(d)$

Theorem 5. Assume $e_n^* = 1$, $f_n^* = 0$,

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$\overline{C}_3(d) := R - R[R + C_0(d)]^{-1}\ R$

Let $c \geq \left(\max_{k=1,\ldots,N} \frac{N}{|\underline{a}_{kk}|^2\,\lambda_k}\right) - \sum_{k=1}^N \frac{1}{|\underline{a}_{kk}|^2\,\lambda_k}$.

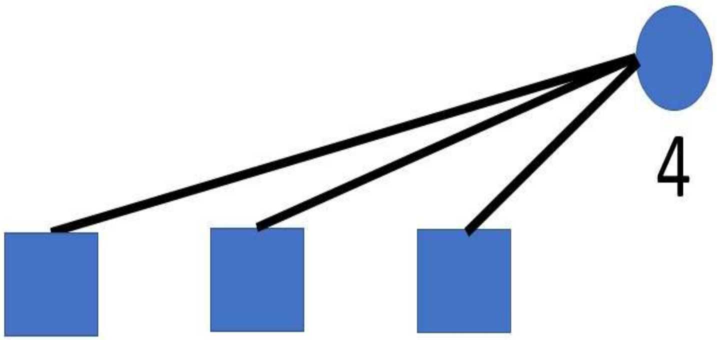
a)

$$\begin{aligned} d_* &= (\delta_1^*, \ldots, \delta_N^*) \in \arg\min\{\det(\overline{C}_3(d));\ d \in \Delta_c\} \\ &\Longleftrightarrow \\ \delta_n^* &= \delta_n^{*c} := \frac{1}{N}\left(c + \sum_{k=1}^N \frac{1}{|\underline{a}_{kk}|^2\,\lambda_k}\right) - \frac{1}{|\underline{a}_{nn}|^2\,\lambda_n}. \end{aligned}$$

b) $\delta_n^{*c} = 0 \Longleftrightarrow c = \frac{N}{|\underline{a}_{nn}|^2\,\lambda_n} - \sum_{k=1}^N \frac{1}{|\underline{a}_{kk}|^2\,\lambda_k}$.

c) $\lim_{c\to\infty} \delta_n^{*c}/\frac{c}{N} = 1$ for all $n = 1, \ldots, N$.

Example and simulation



$\underline{a_{11}} = 30 - j10, \; \underline{a_{22}} = 260 - j100, \; \underline{a_{33}} = 820 - j310.$

$$d_0 = d^*_{\mathsf{tr}(C_0)} = \left(\frac{\frac{1}{|\underline{a_{11}}|}}{\sum_{k=1}^{3} \frac{1}{|\underline{a_{kk}}|}}, \frac{\frac{1}{|\underline{a_{22}}|}}{\sum_{k=1}^{3} \frac{1}{|\underline{a_{kk}}|}}, \frac{\frac{1}{|\underline{a_{33}}|}}{\sum_{k=1}^{3} \frac{1}{|\underline{a_{kk}}|}} \right) = (0.87, 0.10, 0.03),$$

$$d_1 = d^*_{\det(C_0)} = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

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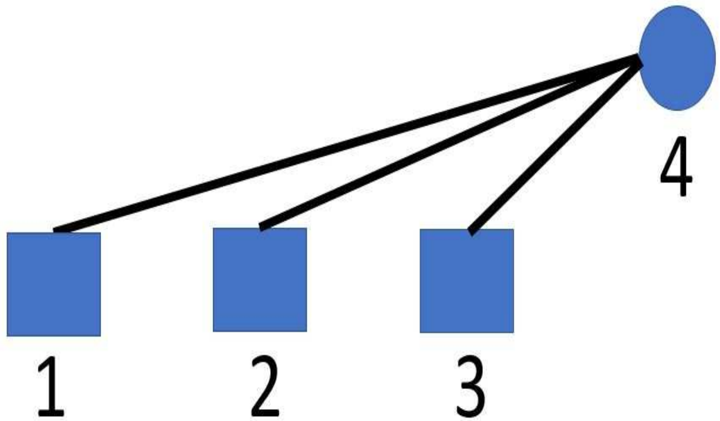

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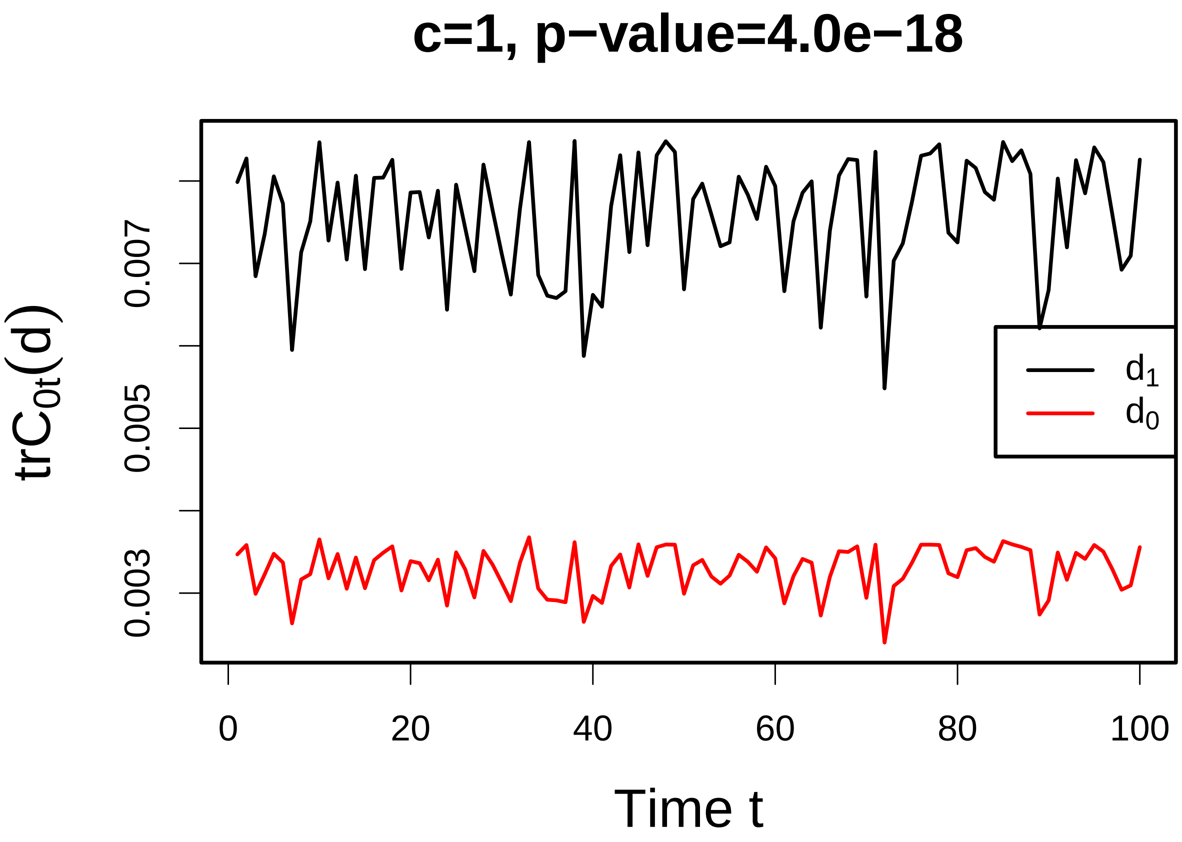
$$d_1 = d^*_{\det(C_0)} = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}).$$

Simulation model for $(\delta_1^*, \delta_2^*, \delta_3^*) = d_0$ or $(\delta_1^*, \delta_2^*, \delta_3^*) = d_1$:

$$V_t = V_{t-1} + 0.9\,(v_0 - V_{t-1}) + E_t^v,\quad E_t^v \sim \mathcal{N}_6(0_6, 0.1\,\mathbb{I}_{6\times 6}),\quad Z_t = h(V_t) + E_t^z,$$
$$E_t^z \sim \mathcal{N}_6\left(0_6, \mathrm{diag}\left((\delta_1^*)^{-1}, (\delta_1^*)^{-1}, (\delta_2^*)^{-1}, (\delta_2^*)^{-1}, (\delta_3^*)^{-1}, (\delta_3^*)^{-1}\right)\right).$$

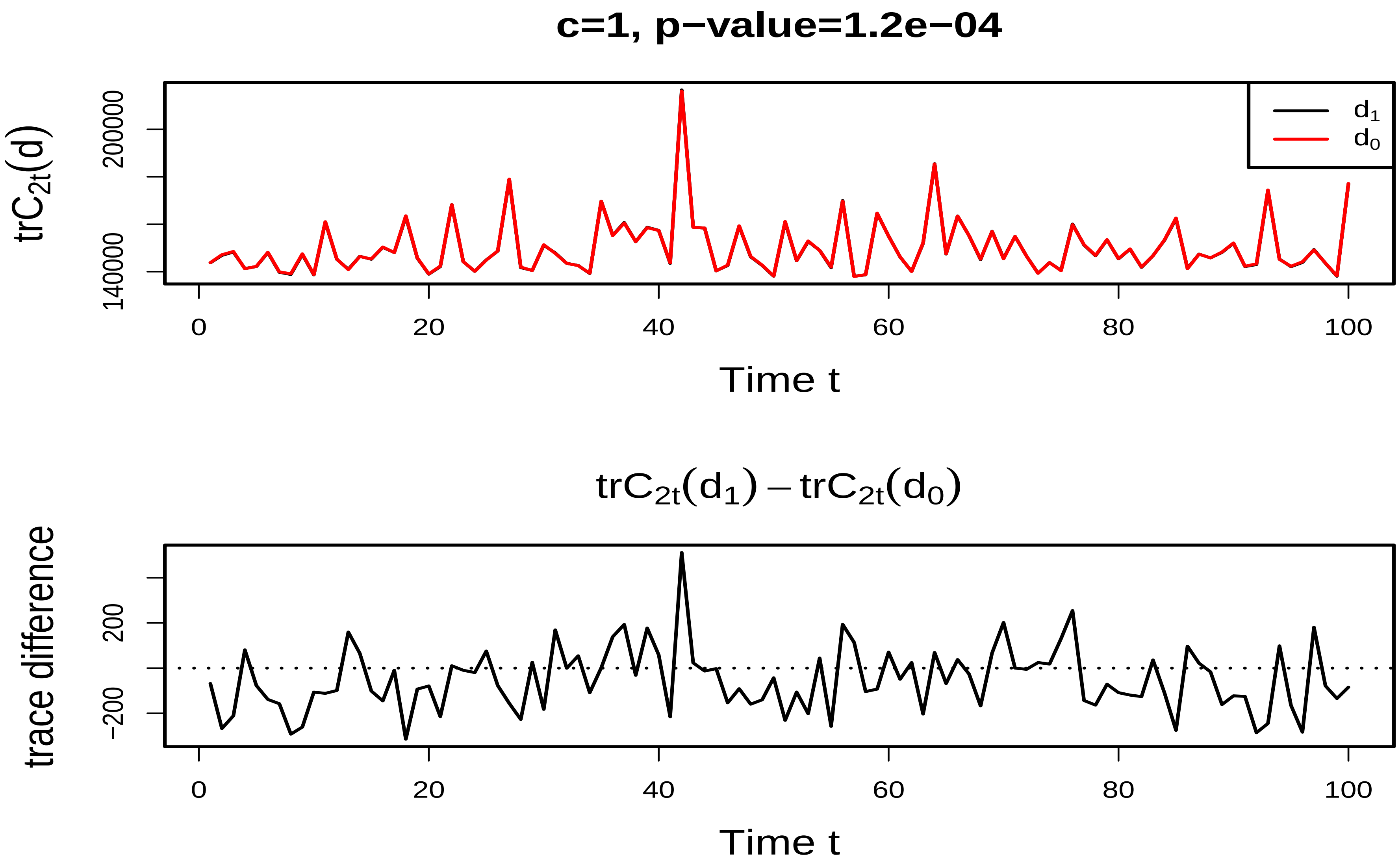
$$C_{0,t}(d) := h'(a_t)^{-1}\Sigma(d)\, h'(a_t)^{-T},\ d_0 \in \arg\min\{\mathrm{tr}(C_0(d);\ d \in \Delta_c\}$$



Complete similar result for $\mathrm{tr}(C_{3,t}(d))$ and very similar result for $\mathrm{tr}(C_{1,t}(d))$.

Simulation

$C_{2,t}(d)$ (Cov for power prediction), $d_0 \in \arg\min\{\text{tr}(C_0(d);\ d \in \Delta_c\}$



Simulation

$d_0 \in \arg\min\{\text{tr}(C_0(d);\ d \in \Delta_c\}$, $d_1 \in \arg\min\{\det(C_0(d);\ d \in \Delta_c\}$

Median differences $\text{tr}(C_{i,t}(d_1)) - \text{tr}(C_{i,t}(d_0))$

c	$\text{tr}(C_{0,t}(d))$	$\text{tr}(C_{1,t}(d))$	$\text{tr}(C_{2,t}(d))$	$\text{tr}(C_{3,t}(d))$
1e+03	4.4e-06	4.3e-08	-0.33	4.4e-06
	(4.0e-18)	(5.8e-18)	(0.72)	(4.0e-18)
1e+01	4.4e-04	4.30e-06	-9.60	4.3e-04
	(4.0e-18)	(5.8e-18)	(0.30)	(4.0e-18)
1e+00	4.4e-03	4.14e-05	-0.77	4.1e-03
	(4.0e-18)	(5.8e-18)	(1.2e-04)	(4.0e-18)
1e-03	4.52	-1.1e-03	-6.7e+04	-0.11
	(4.0e-18)	(5.8e-18)	(4.0e-18)	(4.0e-18)

$d_0 \in \arg\min\{\text{tr}(C_0(d);\ d \in \Delta_c\}$, $d_1 \in \arg\min\{\det(C_0(d);\ d \in \Delta_c\}$

Median differences $\text{tr}(C_{i,t}(d_1)) - \text{tr}(C_{i,t}(d_0))$

c	$\text{tr}(C_{0,t}(d))$	$\text{tr}(C_{1,t}(d))$	$\text{tr}(C_{2,t}(d))$	$\text{tr}(C_{3,t}(d))$
1e+03	4.4e-06	4.3e-08	-0.33	4.4e-06
	(4.0e-18)	(5.8e-18)	(0.72)	(4.0e-18)
1e+01	4.4e-04	4.30e-06	-9.60	4.3e-04
	(4.0e-18)	(5.8e-18)	(0.30)	(4.0e-18)
1e+00	4.4e-03	4.14e-05	-0.77	4.1e-03
	(4.0e-18)	(5.8e-18)	(1.2e-04)	(4.0e-18)
1e-03	4.52	-1.1e-03	-6.7e+04	-0.11
	(4.0e-18)	(5.8e-18)	(4.0e-18)	(4.0e-18)

Recall $C_{2,t}(d) := h'(a_t)R_t(d)\,h'(a_t)^\top + \Sigma(d)$.

Simulation

$d_0 \in \arg\min\{\mathrm{tr}(C_0(d);\ d \in \Delta_c\}$, $d_1 \in \arg\min\{\det(C_0(d);\ d \in \Delta_c\}$

Median differences $\ln\det(C_{i,t}(d_1)) - \ln\det(C_{i,t}(d_0))$

c	$\ln\det(C_{0,t}(d))$	$\ln\det(C_{1,t}(d))$	$\ln\det(C_{2,t}(d))$	$\ln\det(C_{3,t}(d))$
1e+03	-5.24	4.3e-07	4.1e-05	-5.24
	(4.0e-18)	(5.8e-18)	(6.0e-02)	(4.0e-18)
1e+01	-5.24	4.3e-05	4.4e-03	-5.25
	(4.0e-18)	(5.8e-18)	(5.0e-14)	(4.0e-18)
1e+00	-5.24	4.1e-04	4.2e-02	-5.28
	(4.0e-18)	(5.8e-18)	(4.1e-18)	(4.0e-18)
1e-03	-5.30	-1.1e-02	3.6e-03	-5.26
	(4.0e-18)	(5.8e-18)	(0.45)	(4.0e-18)

Simulation

Wilcoxon test applied although the independence of differences is not clear.

Wilcoxon test applied although the independence of differences is not clear.

However, the Ljung-Box test provides

minimum p-values of 0.07 and a mean p-value of 0.68 for lag 1,

minimum p-value of 0.08 and a mean p-value of 0.72 for lag 10.

Similar results for:

- $V_t = V_{t-1} + \Theta_v(v_0 - V_{t-1}) + E_t^v$
  with v_0 as above and Θ_v is a block diagonal matrix of 2×2 blocks,

Similar results for:

- $V_t = V_{t-1} + \Theta_v(v_0 - V_{t-1}) + E_t^v$
  with v_0 as above and Θ_v is a block diagonal matrix of 2×2 blocks,
- $V_t = 0.9\, V_{t-1} + E_t^v$,
- $V_t = \Theta_v V_{t-1} + E_t^v$.

Conclusion:

complete similar behaviour of $\det(C_{0,t}(d))$ and $\det(C_{3,t}(d))$,

similar behaviour of $\mathrm{tr}(C_{0,t}(d))$ and $\mathrm{tr}(C_{3,t}(d))$ except for small c.

Conclusion:

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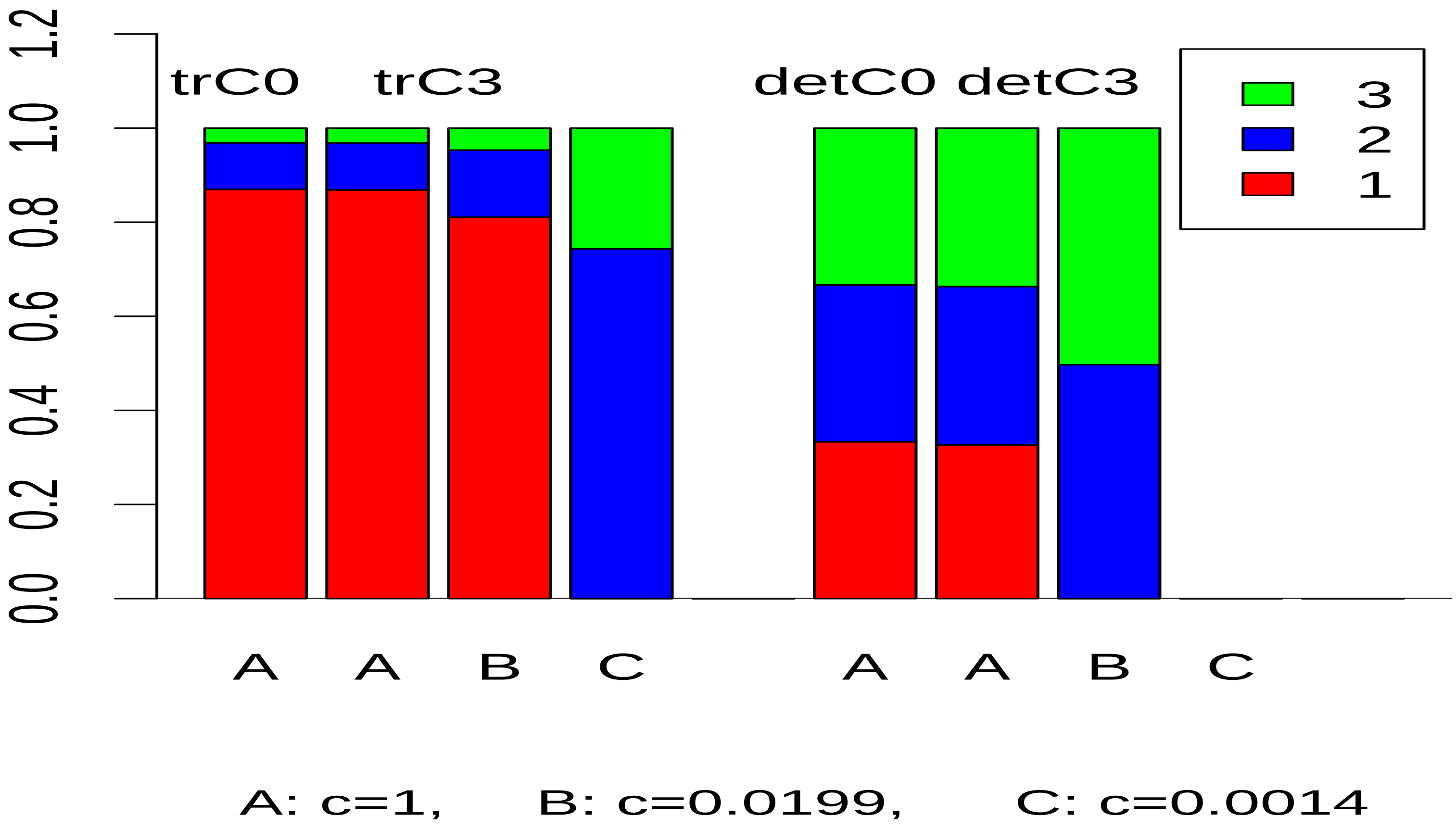
Why the different behaviour of $\text{tr}(C_{3,t}(d))$ for very small c?

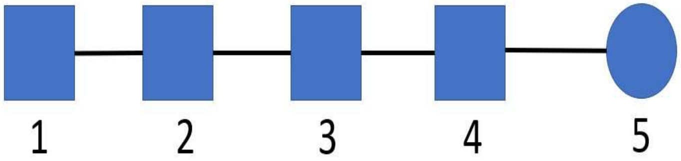
Why the different behaviour of $\text{tr}(C_{3,t}(d))$ for very small c?

A possible explanation can be Theorems 4 and 5:

Proportions of A- and D- optimal design for $C_0(d)$ and $\overline{C}_3(d)$ denoted by C0 and C3:



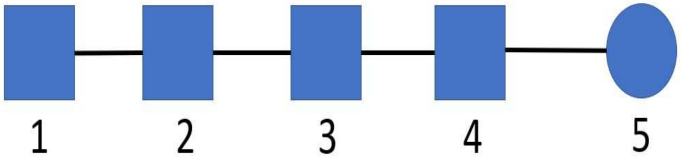
Simulation for series connection



$$\underline{A} =$$

$$\begin{pmatrix} 1.47 - 3.73i & -1.47 + 3.73i & 0.00 + 0.00i & 0.00 + 0.00i & 0.00 + 0.00i \\ -1.47 + 3.72i & 5.28 - 4.92i & -3.81 + 1.20i & 0.00 + 0.00i & 0.00 + 0.00i \\ 0.00 + 0.00i & -3.81 + 1.20i & 5.20 - 1.63i & -1.39 + 0.44i & 0.00 + 0.00i \\ 0.00 + 0.00i & 0.00 + 0.00i & -1.39 + 0.434i & 3.07 - 0.96i & -1.68 + 0.53i \\ 0.00 + 0.00i & 0.00 + 0.00i & 0.00 + 0.00i & -1.68 + 0.53i & 1.68 - 0.53i \end{pmatrix}.$$

Simulation for series connection



$$\underline{A} =$$

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$$\overline{\Delta}_1 =$$
$$\{(0.7,\, 0.1,\, 0.1,\, 0.1), (0.1,\, 0.7,\, 0.1,\, 0.1), (0.1,\, 0.1,\, 0.7,\, 0.1), (0.1,\, 0.1,\, 0.1,\, 0.7)\}$$

$$d_0 = (0.7,\, 0.1,\, 0.1,\, 0.1) = \arg\min\{\mathrm{tr}(C_0(d),\ d \in \overline{\Delta}_1\}$$
$$d_1 = (0.1,\, 0.1,\, 0.1,\, 0.7) = \arg\max\{\mathrm{tr}(C_0(d),\ d \in \overline{\Delta}_1\}$$

Simulation for series connection

$\overline{\Delta}_1 =$
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$d_1 = (0.1,\, 0.1,\, 0.1,\, 0.7) = \arg\max\{\mathrm{tr}(C_0(d),\ d \in \overline{\Delta}_1\}$

Simulation model for $(\delta_1^*, \delta_2^*, \delta_3^*, \delta_3^*, \delta_4^*) = d_0$ or $(\delta_1^*, \delta_2^*, \delta_3^*, \delta_4^*) = d_1$:

$V_t = V_{t-1} + 0.9\,(v_0 - V_{t-1}) + E_t^v,\quad E_t^v \sim \mathcal{N}_8(0_8, 0.1\, I\!I_{8\times 8}),\quad Z_t = h(V_t) + E_t^z,$
$E_t^z \sim$
$\mathcal{N}_8\left(0_8, \mathrm{diag}\left((\delta_1^*)^{-1}, (\delta_1^*)^{-1}, (\delta_2^*)^{-1}, (\delta_2^*)^{-1}, (\delta_3^*)^{-1}, (\delta_3^*)^{-1}, (\delta_4^*)^{-1}, (\delta_4^*)^{-1}\right)\right).$

Simulation for series connection

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$$d_0 = (0.7,\, 0.1,\, 0.1,\, 0.1) = \arg\min\{\mathrm{tr}(C_0(d),\; d \in \overline{\Delta}_1\}$$
$$d_1 = (0.1,\, 0.1,\, 0.1,\, 0.7) = \arg\max\{\mathrm{tr}(C_0(d),\; d \in \overline{\Delta}_1\}$$

Median differences $\mathrm{tr}(C_{i,t}(d_1)) - \mathrm{tr}(C_{i,t}(d_0))$

c	$\mathrm{tr}(C_{0,t}(d))$	$\mathrm{tr}(C_{1,t}(d))$	$\mathrm{tr}(C_{2,t}(d))$	$\mathrm{tr}(C_{3,t}(d))$
1e+02	1.41	-7.5e-05	-0.34	-0.01
	(4.0e-18)	(5.8e-18)	(7.8e-18)	(4.0e-18)
1e+00	1.2e+02	2.1e-05	0.07	0.002
	(4.3e-18)	(4.5e-13)	(2.3e-05)	(2.8e-13)
1e-01	1.2e+03	1.3e-04	0.03	0.01
	(4.0e-18)	(5.8e-18)	(4.0e-04)	(4.0e-18)
1e-03	1.2e+05	2.0e-06	0.002	0.0002
	(4.0e-18)	(5.8e-18)	(9.3e-02)	(4.0e-18)

Simulation for series connection

$$d_0 = (0.7,\ 0.1,\ 0.1,\ 0.1) = \arg\min\{\mathrm{tr}(C_0(d),\ d \in \overline{\Delta}_1\}$$
$$d_1 = (0.1,\ 0.1,\ 0.1,\ 0.7) = \arg\max\{\mathrm{tr}(C_0(d),\ d \in \overline{\Delta}_1\}$$

Median differences $\mathrm{tr}(C_{i,t}(d_1)) - \mathrm{tr}(C_{i,t}(d_0))$

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1e+00	1.2e+02	2.1e-05	0.07	0.002
	(4.3e-18)	(4.5e-13)	(2.3e-05)	(2.8e-13)
1e-01	1.2e+03	1.3e-04	0.03	0.01
	(4.0e-18)	(5.8e-18)	(4.0e-04)	(4.0e-18)
1e-03	1.2e+05	2.0e-06	0.002	0.0002
	(4.0e-18)	(5.8e-18)	(9.3e-02)	(4.0e-18)

Recall $C_{2,t}(d) := h'(a_t)R_t(d)\,h'(a_t)^\top + \Sigma(d)$.

Summary and open problems

Simplified criterion: $C_{0,t}(d) := h'(a_t)^{-1}\Sigma(d)\, h'(a_t)^{-T}$,

$d_0 \in \arg\min\{\text{tr}(C_0(d));\ d \in \Delta\}, \quad d_1 \in \arg\min\{\det(C_{0,t}(d));\ d \in \Delta\}$

Star network:

	Better	
	tr	det
Covariance of voltage prediction $C_{1,t}(d)$	d_0	d_0
Covariance of power prediction $C_{2,t}(d)$	d_1	d_0
Covariance of voltage estimation $C_{3,t}(d)$	d_0	d_1

Summary and open problems

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Covariance of voltage prediction $C_{1,t}(d)$	d_0	d_0
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Covariance of voltage estimation $C_{3,t}(d)$	d_0	d_1

Open Problem

Does these results for the star network and a simple model transfer to more general networks and models?

What are A-optimal designs for series connections?

Is $C_0(d)$ or $C_{0,t}(d)$ always a good criterion for sensor allocation?